Determination of Fricke families

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Abstract

For a positive integer N divisible by 4, let $\mathcal{O}^1_N(\mathbb{Q})$ be the ring of weakly holomorphic modular functions for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients. We construct explicit generators of $\mathcal{O}^1_N(\mathbb{Q})$ over $\mathbb{Q}$, from which we classify all Fricke families of level N.

1 Introduction

For a positive integer N, let $\mathcal{F}_N$ be the field of meromorphic modular functions for the principal congruence subgroup $\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2 \pmod{N}\}$ whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$ where $\zeta_N = e^{2\pi i/N}$. It is well known that $\mathcal{F}_1$ is generated over $\mathbb{Q}$ by the elliptic modular function $j(\tau)$ ($\tau \in \mathbb{H}$, the complex upper half-plane), and $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \tag{1}$$

(see §2). For $N \geq 2$ we let

$$\mathcal{V}_N = \{\mathbf{v} \in \mathbb{Q}^2 \mid \mathbf{v} \text{ has primitive denominator } N\}.$$

We call a family $\{h_\mathbf{v}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ a *Fricke family* of level N if it satisfies the following three conditions:

(F1) Each $h_\mathbf{v}(\tau)$ is weakly holomorphic (that is, holomorphic on $\mathbb{H}$).

(F2) $h_\mathbf{v}(\tau)$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$.

(F3) $h_\mathbf{v}(\tau)^\gamma = h_{{}^t\gamma\mathbf{v}}(\tau)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$, where ${}^t\gamma$ means the transpose of γ.

There are two important kinds of Fricke families $\{f_\mathbf{v}(\tau)\}_\mathbf{v}$ and $\{g_\mathbf{v}(\tau)^{12N}\}_\mathbf{v}$, one consisting of Fricke functions and the other consisting of $12N$th powers of Siegel functions (see §3). They are building blocks of fields of modular functions and groups of modular units ([6, Chapter 2] and

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[8, Chapter 6]). Furthermore, their special values at imaginary quadratic arguments generate class fields over the corresponding imaginary quadratic fields ([3], [4] and [8, Chapter 10]).

In this paper we shall classify all Fricke families of level N when $N \equiv 0 \pmod 4$ (Theorems 4.3, 6.2 and Corollary 6.3). Moreover, if we weaken the condition (F1) as

(F1$'$) every $h_{\mathbf{v}}(\tau)$ is holomorphic on $\mathbb{H}$ except for the set $\{\gamma(\zeta_3), \gamma(\zeta_4) \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$,

then we can also determine all families $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v} \in \mathcal{V}_N}$ of functions in $\mathcal{F}_N$ satisfying (F1$'$), (F2) and (F3) for arbitrary level $N \geq 1$ (Theorem 7.4 and Remark 7.5).

2 Galois actions on functions

In this section we shall briefly describe the actions of the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ on the field $\mathcal{F}_N$ based on [9, §6.1–6.2].

For a positive integer N, $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ has a unique decomposition

$$G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \quad \text{with} \quad G_N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

This group acts on the field $\mathcal{F}_N$ as follows ([9, §6.1–6.2]): Let

$$h(\tau) = \sum_{n \gg -\infty} c_n q^{n/N} \in \mathcal{F}_N \quad (c_n \in \mathbb{Q}(\zeta_N),\ q = e^{2\pi i \tau}).$$

(A1) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \in G_N$ acts on $h(\tau)$ as

$$h(\tau)^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right]} = \sum_{n \gg -\infty} c_n^{\sigma_d} q^{n/N},$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ given by $\zeta_N^{\sigma_d} = \zeta_N^d$.

(A2) The matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $h(\tau)$ as

$$h(\tau)^\gamma = (h \circ \widetilde{\gamma})(\tau),$$

where $\widetilde{\gamma}$ is any preimage of the reduction $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$.

Lemma 2.1. *Let $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v} \in \mathcal{V}_N}$ be a Fricke family of level $N \geq 2$. Then, $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ transitively.*

Proof. Note by (F3) that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on the family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$. Let $\mathbf{v} = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_N$, so $\gcd(a, b)$ is relatively prime to N. If we take any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ such that $\det(\gamma)$ is relatively prime to N, then we find by (F3) that

$$h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^\gamma = h_{{}^t\gamma\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) = h_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) = h_{\mathbf{v}}(\tau).$$

This implies that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ transitively. $\square$

Remark 2.2. Roughly speaking, $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ is completely determined by its component $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$.

3 Fricke and Siegel functions

For a lattice Λ in $\mathbb{C}$, we let

$$g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4}, \quad g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^6} \quad \text{and} \quad \Delta(\Lambda) = g_2(\Lambda)^3 - 27 g_3(\Lambda)^2.$$

The *elliptic modular function* $j(\tau)$ is defined by

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)} = 1728 \left(1 + 27 \frac{g_3(\tau)^2}{\Delta(\tau)}\right) \quad (\tau \in \mathbb{H}), \tag{2}$$

where $g_2(\tau) = g_2([\tau, 1])$, $g_3(\tau) = g_3([\tau, 1])$ and $\Delta(\tau) = \Delta([\tau, 1])$. This generates the ring of weakly holomorphic functions in $\mathcal{F}_1$ over $\mathbb{Q}$ ([8, Theorem 2 in Chapter 5]).

The *Weierstrass $\wp$-function* relative to Λ is given by

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right) \quad (z \in \mathbb{C}).$$

For each $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we define the *Fricke function* $f_{\mathbf{v}}(\tau)$ by

$$f_{\mathbf{v}}(\tau) = -2^7 3^5 \frac{g_2(\tau) g_3(\tau)}{\Delta(\tau)} \wp_{\mathbf{v}}(\tau) \quad (\tau \in \mathbb{H}), \tag{3}$$

where $\wp_{\mathbf{v}}(\tau) = \wp(v_1 \tau + v_2; [\tau, 1])$.

Furthermore, the *Weierstrass σ-function* relative to Λ is given by

$$\sigma(z; \Lambda) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda + (1/2)(z/\lambda)^2} \quad (z \in \mathbb{C}).$$

Taking logarithmic derivative yields the *Weierstrass ζ-function* as

$$\zeta(z; \Lambda) = \frac{\sigma'(z; \Lambda)}{\sigma(z; \Lambda)} = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) \quad (z \in \mathbb{C}).$$

Since $\zeta'(z; \Lambda) = -\wp(z; \Lambda)$ is periodic with respect to Λ, for each $\lambda \in \Lambda$ there is a constant $\eta(\lambda; \Lambda)$ so that

$$\zeta(z + \lambda; \Lambda) - \zeta(z; \Lambda) = \eta(\lambda; \Lambda) \quad (z \in \mathbb{C}).$$

For each $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we then define the *Siegel function* $g_{\mathbf{v}}(\tau)$ by

$$g_{\mathbf{v}}(\tau) = e^{-(v_1 \eta(\tau; [\tau, 1]) + v_2 \eta(1; [\tau, 1]))(v_1 \tau + v_2)/2} \sigma(v_1 \tau + v_2; [\tau, 1]) \eta(\tau)^2 \quad (\tau \in \mathbb{H}), \tag{4}$$

where

$$\eta(\tau) = \sqrt{2\pi} \zeta_8 q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (\tau \in \mathbb{H})$$

is the *Dedekind η-function* which is a 24th root of $\Delta(\tau)$ ([8, Theorem 5 in Chapter 18]). By the q-product expansion of the Weierstrass σ-function we get the expression

$$g_{\mathbf{v}}(\tau) = -e^{\pi i v_2(v_1-1)} q^{(1/2)\mathbf{B}_2(v_1)} (1 - q^{v_1} e^{2\pi i v_2}) \prod_{n=1}^{\infty} (1 - q^{n+v_1} e^{2\pi i v_2})(1 - q^{n-v_1} e^{-2\pi i v_2}),$$

where $\mathbf{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial ([8, Chapter 19, §2]). Note that $g_{\mathbf{v}}(\tau)$ has neither zeros nor poles on $\mathbb{H}$.

PROPOSITION 3.1. *If $N \geq 2$, then $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ and $\{g_{\mathbf{v}}(\tau)^{12N}\}_{\mathbf{v}\in\mathcal{V}_N}$ are Fricke families of level N.*

PROOF. See [8, Chapter 6, §2–3] and [6, Proposition 1.3 in Chapter 2]. $\square$

REMARK 3.2. We call a function $h(\tau)$ in $\mathcal{F}_N$ a *modular unit* of level $N \geq 1$ if both $h(\tau)$ and $h(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$. As is well known, $h(\tau)$ is a modular unit if and only if it has neither zeros nor poles on $\mathbb{H}$ ([6, p. 36] or [2, Proposition 2.3]). Thus $g_{\mathbf{v}}(\tau)^{12N}$ is a modular unit of level N for every $\mathbf{v} \in \mathcal{V}_N$ with $N \geq 2$. Moreover, $g_{\mathbf{v}}(\tau)$ is a modular unit of level $12N^2$ ([6, Theorems 5.2 and 5.3 in Chapter 3]).

For later use, we need the following lemma.

LEMMA 3.3. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$.*

(i) *We have the assertion that $f_{\mathbf{u}}(\tau) = f_{\mathbf{v}}(\tau)$ if and only if $\mathbf{u} \equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$.*

(ii) *If $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$, then we get the relation*

$$(f_{\mathbf{u}}(\tau) - f_{\mathbf{v}}(\tau))^6 = 2^{12}3^6 j(\tau)^2 (j(\tau) - 1728)^3 \frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12} g_{\mathbf{v}}(\tau)^{12}}.$$

(iii) *For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we get $(g_{\mathbf{v}} \circ \gamma)(\tau) = \zeta g_{{}^t\gamma\mathbf{v}}(\tau)$ for some 12th root of unity ζ depending only on γ.*

PROOF. (i) See [1, Lemma 10.4] and definition (3).
(ii) See [8, Theorem 2 in Chapter 18] and definitions (2), (3) and (4).
(iii) See [7, Proposition 2.4]. $\square$

4 Rings of weakly holomorphic functions

For an integer $N \geq 2$, we denote by Fr_N the set of all Fricke families of level N. Then, Fr_N becomes a ring under the operations

$$\begin{aligned} \{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} + \{k_{\mathbf{v}}(\tau)\}_{\mathbf{v}} &= \{(h_{\mathbf{v}} + k_{\mathbf{v}})(\tau)\}_{\mathbf{v}}, \\ \{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} \cdot \{k_{\mathbf{v}}(\tau)\}_{\mathbf{v}} &= \{(h_{\mathbf{v}} k_{\mathbf{v}})(\tau)\}_{\mathbf{v}}. \end{aligned} \tag{5}$$

For a positive integer N, let $\mathcal{F}_N^1(\mathbb{Q})$ be the field of meromorphic modular functions for the congruence subgroup

$$\Gamma^1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \pmod{N} \right\}$$

with rational Fourier coefficients. Furthermore, we let $\mathcal{O}_N^1(\mathbb{Q})$ be its subring consisting of weakly holomorphic functions.

LEMMA 4.1. *Let $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} \in \mathrm{Fr}_N$ with $N \geq 2$. Then, $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ belongs to $\mathcal{O}_N^1(\mathbb{Q})$.*

PROOF. For any $\gamma \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^1(N)$, we find that

$$\begin{aligned} h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{\gamma} &= h_{{}^t\gamma\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) \quad \text{by (F3)} \\ &= h_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) \\ &= h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) \quad \text{by the fact } a \equiv 1,\ b \equiv 0 \pmod{N} \text{ and (F2).} \end{aligned}$$

Thus $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ is modular for $\Gamma^1(N)$.

Now, let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \in G_N$. We get by (F3) and (F2) that

$$h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{\alpha} = h_{{}^t\alpha\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) = h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau),$$

which shows that $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ has rational Fourier coefficients by (A1).

Moreover, since $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ is weakly holomorphic by (F1), it belongs to $\mathcal{O}_N^1(\mathbb{Q})$. □

Thus we obtain by Lemma 4.1 a ring homomorphism

$$\begin{array}{rccc} \phi_N : & \mathrm{Fr}_N & \rightarrow & \mathcal{O}_N^1(\mathbb{Q}) \\ & \{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} & \mapsto & h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau). \end{array} \tag{6}$$

LEMMA 4.2. *Let $N \geq 2$, and let a and b be a pair of integers such that $\gcd(a, b)$ is relatively prime to N. Let $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\beta' = \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}$ be matrices in $M_2(\mathbb{Z})$ such that $\det(\beta) \equiv \det(\beta') \equiv 1 \pmod{N}$. Then, there is a matrix $\alpha \in \Gamma^1(N)$ so that $\alpha\beta \equiv \beta' \pmod{N}$.*

PROOF. Take $\alpha = \begin{bmatrix} 1 & 0 \\ c'd - cd' & 1 \end{bmatrix} \in \Gamma^1(N)$. One can readily show that

$$\alpha\beta \equiv \begin{bmatrix} a & b \\ c' \det(\gamma) + c(-\det(\gamma') + 1) & d' \det(\gamma) + d(-\det(\gamma') + 1) \end{bmatrix} \equiv \beta' \pmod{N}$$

due to the fact $\det(\beta) \equiv \det(\beta') \equiv 1 \pmod{N}$. □

Theorem 4.3. *If $N \geq 2$, then* Fr_N *and* $\mathcal{O}_N^1(\mathbb{Q})$ *are isomorphic via the map* ϕ_N *given in* (6).

Proof. Let $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} \in \ker(\phi)$, so $\phi_N(\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}) = h_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) = 0$. Then we get by Lemma 2.1 that $h_{\mathbf{v}}(\tau) = 0$ for all $\mathbf{v} \in \mathcal{V}_N$. This shows that ϕ_N is one-to-one.

Now, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. For each $\mathbf{v} = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_N$, we take any $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ such that $\det(\beta) \equiv 1 \pmod N$, and set $h_{\mathbf{v}}(\tau) = h(\tau)^{\beta}$. We claim that $h_{\mathbf{v}}(\tau)$ is well-defined, independent of the choice of β. Indeed, if $\beta' = \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}$ is another matrix in $M_2(\mathbb{Z})$ such that $\det(\beta') \equiv 1 \pmod N$, then we see that

$$\begin{aligned} h(\tau)^{\beta'} &= h(\tau)^{\alpha\beta} \quad \text{for some } \alpha \in \Gamma^1(N) \text{ by Lemma 4.2 and (1)} \\ &= h(\tau)^{\beta} \quad \text{since } h(\tau) \text{ is modular for } \Gamma^1(N). \end{aligned}$$

Since $h(\tau)$ is weakly holomorphic, so is $h_{\mathbf{v}}(\tau) = h(\tau)^{\beta}$ by (A2). Furthermore, $h_{\mathbf{v}}(\tau)$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$ by (1). Let $\gamma = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. We derive by considering $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as an element of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ that

$$\begin{aligned} h_{\mathbf{v}}(\tau)^{\gamma} &= \left(h(\tau)^{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}\right)^{\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]} \\ &= h(\tau)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{smallmatrix}\right]} \\ &= \left(h(\tau)^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & \det(\gamma) \end{smallmatrix}\right]}\right)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\gamma)^{-1}(cx+dz) & \det(\gamma)^{-1}(cy+dw) \end{smallmatrix}\right]} \\ &= h(\tau)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\gamma)^{-1}(cx+dz) & \det(\gamma)^{-1}(cy+dw) \end{smallmatrix}\right]} \quad \text{since } h(\tau) \text{ has rational Fourier coefficients} \\ &= h_{\left[\begin{smallmatrix} (ax+bz)/N \\ (ay+bw)/N \end{smallmatrix}\right]}(\tau) \quad \text{because } \left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\gamma)^{-1}(cx+dz) & \det(\gamma)^{-1}(cy+dw) \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \\ &= h_{\left[\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}\right]\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) \\ &= h_{{}^t\gamma\mathbf{v}}(\tau). \end{aligned}$$

Thus the family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ satisfies (F3). Lastly, since

$$\phi_N(\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}) = h_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau),$$

ϕ_N is surjective.

Therefore, we conclude that Fr_N and $\mathcal{O}_N^1(\mathbb{Q})$ are isomorphic via the map ϕ_N. $\square$

5 Conjugate subgroups of $\mathrm{SL}_2(\mathbb{R})$

For a positive integer N, let

$$\Gamma_1(N)=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\mid\gamma\equiv\begin{bmatrix}1&*\\0&1\end{bmatrix}\pmod N\right\}\quad\text{and}\quad\omega_N=\begin{bmatrix}1/\sqrt{N}&0\\0&\sqrt{N}\end{bmatrix}.$$

From the observation

$$\omega_N\begin{bmatrix}a&b\\c&d\end{bmatrix}\omega_N^{-1}=\begin{bmatrix}a&b/N\\Nc&d\end{bmatrix}\ \text{for all}\ \begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{SL}_2(\mathbb{R}),$$

we note that $\Gamma^1(N)$ and $\Gamma_1(N)$ are conjugate in $\mathrm{SL}_2(\mathbb{R})$, namely

$$\omega_N\Gamma^1(N)\omega_N^{-1}=\Gamma_1(N). \tag{7}$$

Let $\mathcal{F}_{1,N}(\mathbb{Q})$ be the field of meromorphic modular functions for $X_1(N)$ with rational Fourier coefficients. One can readily check that the relation (7) gives rise to an isomorphism

$$\begin{array}{rcl}\mathcal{F}_{1,N}(\mathbb{Q}) & \stackrel{\sim}{\to} & \mathcal{F}_N^1(\mathbb{Q})\\ h(\tau)=\sum_{n\gg-\infty}c_nq^n & \mapsto & (h\circ\omega_N)(\tau)=h(\tau/N)=\sum_{n\gg-\infty}c_nq^{n/N}\end{array} \tag{8}$$

with inverse map $f(\tau)\mapsto(f\circ\omega_N^{-1})(\tau)=f(N\tau)$. Furthermore, we let $\mathcal{O}_{1,N}(\mathbb{Q})$ be the subring of $\mathcal{F}_{1,N}(\mathbb{Q})$ consisting of weakly holomorphic functions. Since the map in (8) preserves weakly holomorphicity, it induces an isomorphism

$$\mathcal{O}_{1,N}(\mathbb{Q})\stackrel{\sim}{\to}\mathcal{O}_N^1(\mathbb{Q}). \tag{9}$$

Let $X_1(4)$ be the modular curve corresponding to the group $\Gamma_1(4)$. It is well known that $X_1(4)$ is of genus 0 with three inequivalent cusps 0, 1/2 and $i\infty$ ([5, p. 131]). Moreover, the function

$$g_{1,4}(\tau)=\left(\frac{g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(4\tau)}{g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(4\tau)}\right)^8=q^{-1}(1+q)^8\prod_{n=1}^{\infty}\left(\frac{(1-q^{4n+2})(1-q^{4n-2})}{(1-q^{4n+1})(1-q^{4n-1})}\right)^8$$

generates the function field $\mathbb{C}(X_1(4))$ of $X_1(4)$ over $\mathbb{C}$, having values 16, 0 and ∞ at cusps 0, 1/2 and $i\infty$, respectively ([5, Theorem 3 (ii)] and [7, Tables 2 and 3]). Since $g_{1,4}(\tau)$ has rational Fourier coefficients, we obtain by [5, Lemma 4.1] that

$$\mathcal{F}_{1,4}(\mathbb{Q})=\mathbb{Q}(g_{1,4}(\tau)). \tag{10}$$

Lemma 5.1. *Let $c\in\mathbb{C}$. Then, $g_{1,4}(\tau)-c$ has neither zeros nor poles on $\mathbb{H}$ if and only if $c\in\{0,16\}$.*

Proof. See [2, (4)]. □

Theorem 5.2. *We get the following structures.*

(i) $\mathcal{O}_{1,4}(\mathbb{Q}) = \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.

(ii) $\mathcal{O}_4^1(\mathbb{Q}) = \mathbb{Q}[g_4^1(\tau), g_4^1(\tau)^{-1}, (g_4^1(\tau) - 16)^{-1}]$, *where* $g_4^1(\tau) = g_{1,4}(\tau/4) = g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau)^{-8} g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^{8}$.

Proof. (i) Since $g_{1,4}(\tau)$ and $(g_{1,4}(\tau) - 16)$ are modular units in $\mathcal{F}_{1,4}(\mathbb{Q})$ by Lemma 5.1 and (10), we get the inclusion $\mathcal{O}_{1,4}(\mathbb{Q}) \supseteq \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.

Conversely, let $h(\tau) \in \mathcal{O}_{1,4}(\mathbb{Q})$. By (10) we can express $h(\tau)$ as $h(\tau) = A(g_{1,4}(\tau))/B(g_{1,4}(\tau))$ for some polynomials $A(x), B(x) \in \mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c \in \overline{\mathbb{Q}}$ not equal to 0 or 16. We note by Lemma 5.1 that $g_{1,4}(\tau_0) - c = 0$ for some $\tau_0 \in \mathbb{H}$, from which it follows that $B(g_{1,4}(\tau_0)) = 0$. But since $A(x)$ is not divisible by $(x - c)$ in $\overline{\mathbb{Q}}[x]$, we get $A(g_{1,4}(\tau_0)) \neq 0$. This contradicts that $h(\tau)$ is weakly holomorphic. Thus $B(x)$ has no zeros other than 0 and 16, which implies the converse inclusion $\mathcal{O}_{1,4}(\mathbb{Q}) \subseteq \mathbb{Q}[g_{1,4}(\tau), g_{1,4}(\tau)^{-1}, (g_{1,4}(\tau) - 16)^{-1}]$.
(ii) This follows immediately from (i) and the isomorphism given in (9). $\square$

6 Generators for $N \equiv 0 \pmod 4$

Now we shall present explicit generators of the ring $\mathcal{O}_N^1(\mathbb{Q})$ over $\mathbb{Q}$ when $N \equiv 0 \pmod 4$. This amounts to classifying all Fricke families of level N by Theorem 4.3.

Proposition 6.1. *If* $N \geq 2$, *then we get* $\mathcal{F}_N^1(\mathbb{Q}) = \mathcal{F}_1(f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau))$.

Proof. We recall that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N(\mathbb{Z})/\{\pm I_2\} \simeq G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$

Note by (A1) and (A2) that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_N^1(\mathbb{Q})$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \simeq G_N \cdot \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \mid \gamma \equiv \pm \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \pmod N \right\}.$$

Let $F = \mathcal{F}_1(f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau))$. Since $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v} \in \mathcal{V}_N} \in \mathrm{Fr}_N$ by Proposition 3.1, we get the inclusion $F \subseteq \mathcal{F}_N^1(\mathbb{Q})$ by Lemma 4.1. Suppose that $\gamma = \alpha\beta$ with $\alpha \in G_N$ and $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ leaves $f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)$ fixed. We then derive that

$$\begin{aligned} f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) &= (f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)^{\alpha})^{\beta} \\ &= f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)^{\beta} \quad \text{since } f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) \text{ has rational Fourier coefficients} \\ &= f_{{}^t\beta\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) \quad \text{by (F2) and (F3)} \\ &= f_{\left[\begin{smallmatrix}a/N\\b/N\end{smallmatrix}\right]}(\tau). \end{aligned}$$

Thus we get $b \equiv 0 \pmod N$ and $a \equiv d \equiv \pm 1 \pmod N$ by Lemma 3.3 (i) and the fact $\beta \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. This argument implies $F \supseteq \mathcal{F}_N^1(\mathbb{Q})$ by Galois theory. Therefore, we conclude that $F = \mathcal{F}_1(f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)) = \mathcal{F}_N^1(\mathbb{Q})$. $\square$

When $N \geq 8$ and $N \equiv 0 \pmod 4$, we consider a function

$$f_N^1(\tau) = \frac{f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)} \quad (\tau \in \mathbb{H}).$$

It is a modular unit belonging to $\mathcal{O}_N^1(\mathbb{Q})$ by Proposition 3.1, Lemmas 3.3 (i) and 4.1.

THEOREM 6.2. *If $N \geq 8$ and $N \equiv 0 \pmod 4$, then we have*

$$\mathcal{O}_N^1(\mathbb{Q}) = \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)] = \mathbb{Q}[g_4^1(\tau), g_4^1(\tau)^{-1}, (g_4^1(\tau) - 16)^{-1}, f_N^1(\tau)].$$

PROOF. It is obvious that $\mathcal{O}_N^1(\mathbb{Q}) \supseteq \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$.

For the converse inclusion, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. Note by Proposition 6.1 and Lemma 4.1 that

$$\mathcal{F}_N^1(\mathbb{Q}) = \mathcal{F}_1(f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)) = \mathcal{F}_4^1(\mathbb{Q})(f_N^1(\tau)).$$

So we can express $h = h(\tau)$ as

$$h = c_0 + c_1 f + \cdots + c_{d-1} f^{d-1} \tag{11}$$

where $d = [\mathcal{F}_N^1(\mathbb{Q}) : \mathcal{F}_4^1(\mathbb{Q})]$ and $c_0, c_1, \ldots, c_{d-1} \in \mathcal{F}_4^1(\mathbb{Q})$. Multiplying both sides of (11) by $1, f, \ldots, f^{d-1}$, respectively, yields a linear system (with unknowns $c_0, c_1, \ldots, c_{d-1}$)

$$\begin{bmatrix} 1 & f & \cdots & f^{d-1} \\ f & f^2 & \cdots & f^d \\ \vdots & \vdots & \ddots & \vdots \\ f^{d-1} & f^d & \cdots & f^{2d-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} h \\ fh \\ \vdots \\ f^{d-1}h \end{bmatrix}.$$

By taking the trace $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{F}_N^1(\mathbb{Q})/\mathcal{F}_4^1(\mathbb{Q})}$ on both sides we obtain

$$T \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(h) \\ \mathrm{Tr}(fh) \\ \vdots \\ \mathrm{Tr}(f^{d-1}h) \end{bmatrix} \quad \text{with} \quad T = \begin{bmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(f) & \cdots & \mathrm{Tr}(f^{d-1}) \\ \mathrm{Tr}(f) & \mathrm{Tr}(f^2) & \cdots & \mathrm{Tr}(f^d) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}(f^{d-1}) & \mathrm{Tr}(f^d) & \cdots & \mathrm{Tr}(f^{2d-2}) \end{bmatrix}.$$

Since every $\mathrm{Tr}(*)$, appeared in the above expression, lies in $\mathcal{O}_4^1(\mathbb{Q})$, we get

$$c_0, c_1, \ldots, c_{d-1} \in \det(T)^{-1} \mathcal{O}_4^1(\mathbb{Q}). \tag{12}$$

If we let $f_1, f_2, \ldots, f_d$ be all the Galois conjugates of f over $\mathcal{F}_4^1(\mathbb{Q})$, then we find that

$$\det(T) = \begin{vmatrix} \sum_{k=1}^d f_k^0 & \sum_{k=1}^d f_k^1 & \cdots & \sum_{k=1}^d f_k^{d-1} \\ \sum_{k=1}^d f_k^1 & \sum_{k=1}^d f_k^2 & \cdots & \sum_{k=1}^d f_k^d \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^d f_k^{d-1} & \sum_{k=1}^d f_k^d & \cdots & \sum_{k=1}^d f_k^{2d-2} \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} f_1^0 & f_2^0 & \cdots & f_d^0 \\ f_1^1 & f_2^1 & \cdots & f_d^1 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{d-1} & f_2^{d-1} & \cdots & f_d^{d-1} \end{vmatrix} \begin{vmatrix} f_1^0 & f_1^1 & \cdots & f_1^{d-1} \\ f_2^0 & f_2^1 & \cdots & f_2^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_d^0 & f_d^1 & \cdots & f_d^{d-1} \end{vmatrix} \\
&= \prod_{1\le m<n\le d} (f_m - f_n)^2 \quad \text{by the Vandermonde determinant formula.}
\end{aligned}
$$

On the other hand, since $f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)$ and $f_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau)$ belong to $\mathcal{F}_4^1(\mathbb{Q})$, each $(f_m - f_n)$ is of the form

$$
f_m - f_n = \frac{f_{\left[\begin{smallmatrix}a/N\\b/N\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)} - \frac{f_{\left[\begin{smallmatrix}c/N\\d/N\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)} = \frac{f_{\left[\begin{smallmatrix}a/N\\b/N\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}c/N\\d/N\end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)}
$$

for some $\begin{bmatrix} a/N \\ b/N \end{bmatrix}, \begin{bmatrix} c/N \\ d/N \end{bmatrix} \in \mathcal{V}_N$ such that $\begin{bmatrix} a/N \\ b/N \end{bmatrix} \not\equiv \pm \begin{bmatrix} c/N \\ d/N \end{bmatrix} \pmod{\mathbb{Z}^2}$. Thus $\det(T)$ is a modular unit in $\mathcal{O}_4^1(\mathbb{Q})$ by Lemma 3.3 (i), from which it follows by (11) and (12) that $h(\tau) \in \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$. Therefore we achieve the inclusion $\mathcal{O}_N^1(\mathbb{Q}) \subseteq \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$, as desired. □

Corollary 6.3. *Let $N \geq 8$ and $N \geq 0 \pmod 4$. For each $\mathbf{v} = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_N$, let*

$$
r_{\mathbf{v}}(\tau) = \left(\frac{g_{(N/2)\mathbf{v}}(\tau)}{g_{(N/4)\mathbf{v}}(\tau)} \right)^8 \quad \text{and} \quad s_{\mathbf{v}}(\tau) = \frac{f_{\mathbf{v}}(\tau) - f_{(N/2)\mathbf{v}}(\tau)}{f_{(N/4)\mathbf{v}}(\tau) - f_{(N/2)\mathbf{v}}(\tau)}.
$$

Then, a family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ is a Fricke family of level N if and only if there is a polynomial $P(x, y, z, w) \in \mathbb{Q}[x, y, z, w]$ so that

$$
h_{\mathbf{v}}(\tau) = P(r_{\mathbf{v}}(\tau), r_{\mathbf{v}}(\tau)^{-1}, (r_{\mathbf{v}}(\tau) - 16)^{-1}, s_{\mathbf{v}}(\tau)) \quad \text{for all } \mathbf{v} \in \mathcal{V}_N.
$$

Proof. Take any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ such that $\det(\gamma) \equiv 1 \pmod N$. We find by (A1), Lemma 3.3 (iii) and Proposition 3.1 that

$$
r_{\mathbf{v}}(\tau) = g_4^1(\tau)^\gamma \quad \text{and} \quad s_{\mathbf{v}}(\tau) = f_N^1(\tau)^\gamma.
$$

Now, the result follows from Theorems 4.3 (with its proof) and 6.2. □

7 Weak Fricke families

Let $\mathbb{H}' = \mathbb{H} \setminus \{\gamma(\zeta_3), \gamma(\zeta_4) \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$. For a positive integer N, let $\mathcal{O}_N^{1\,\prime}(\mathbb{Q})$ be the ring of functions in $\mathcal{F}_N^1(\mathbb{Q})$ which are holomorphic on $\mathbb{H}'$.

Lemma 7.1. *$j(\tau)$ gives to rise a bijection $j(\tau) : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ such that $j(\zeta_3) = 0$ and $j(\zeta_4) = 1728$.*

Proof. See [8, Theorem 4 in Chapter 3]. $\square$

Theorem 7.2. *We have* $\mathcal{O}_1^{1\prime}(\mathbb{Q}) = \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}]$.

Proof. By Lemma 7.1 we get the inclusion $\mathcal{O}_1^{1\prime}(\mathbb{Q}) \supseteq \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}]$.

Now, let $h(\tau) \in \mathcal{O}_1^{1\prime}(\mathbb{Q})$. Since $\mathcal{F}_1^1(\mathbb{Q}) = \mathcal{F}_1 = \mathbb{Q}(j(\tau))$, we have $h(\tau) = A(j(\tau))/B(j(\tau))$ for some polynomials $A(x), B(x) \in \mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c \in \overline{\mathbb{Q}}$ not equal to 0 and 1728. Since $j(\tau_0) = c$ for some $\tau_0 \in \mathbb{H}'$ by Lemma 7.1, we have $B(j(\tau_0)) = 0$. But since $A(x)$ is not divisible by $(x-c)$, we see that $A(j(\tau_0)) \neq 0$, which contradicts that $h(\tau)$ is holomorphic on $\mathbb{H}'$. Thus we conclude that 0 and 1728 are the only possible zeros of $B(x)$. This proves the converse inclusion $\mathcal{O}_1^{1\prime}(\mathbb{Q}) \subseteq \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}]$. $\square$

Lemma 7.3. *Modular units of level* 1 *are exactly nonzero rational numbers.*

Proof. See [7, Lemma 2.1]. One can also give a proof by using Lemma 7.1. $\square$

Theorem 7.4. *If* $N \geq 2$, *then we obtain*

$$\mathcal{O}_N^{1\prime}(\mathbb{Q}) = \mathcal{O}_1^{1\prime}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)] = \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}, f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)].$$

Proof. Since $f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)$ is weakly holomorphic, we get the inclusion $\mathcal{O}_N^{1\prime}(\mathbb{Q}) \supseteq \mathcal{O}_1^{1\prime}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$.

For the converse inclusion, let $h(\tau) \in \mathcal{O}_N^{1\prime}(\mathbb{Q})$. Since $\mathcal{F}_N^1(\mathbb{Q})$ is generated by $f = f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)$ over $\mathcal{F}_1 = \mathcal{F}_1^1(\mathbb{Q})$ by Proposition 6.1, we can write

$$h = c_0 + c_1 f + \cdots + c_{d-1} f^{d-1} \tag{13}$$

where $d = [\mathcal{F}_N^1(\mathbb{Q}) : \mathcal{F}_1^1(\mathbb{Q})]$ and $c_0, c_1, \ldots, c_{d-1} \in \mathcal{F}_1^1(\mathbb{Q})$. If $f_1, f_2, \ldots, f_d$ denotes all the Galois conjugates of f over $\mathcal{F}_1^1(\mathbb{Q})$, and $D = \prod_{1\leq m,n\leq d}(f_m - f_n)^2$, then one can show that

$$c_0, c_1, \ldots, c_{d-1} \in D^{-1}\mathcal{O}_1^{1\prime}(\mathbb{Q}). \tag{14}$$

as in the proof of Theorem 6.2. By Lemma 3.3 (ii) we see that each $(f_m - f_n)^6$ is of the form

$$(f_m - f_n)^6 = 2^{12}3^6 j(\tau)^2 (j(\tau)-1728)^3 \frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12} g_{\mathbf{v}}(\tau)^{12}}$$

for some $\mathbf{u}, \mathbf{v} \in \mathcal{V}_N$ such that $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$. It follows by Lemma 7.3 that

$$D = c j(\tau)^{d(d-1)/3}(j(\tau)-1728)^{d(d-1)/2} \quad \text{for some nonzero } c \in \mathbb{C}.$$

Since $D \in \mathcal{F}_1^1(\mathbb{Q}) = \mathbb{Q}(j(\tau))$, we must have $d(d-1)/3 \in \mathbb{Z}$ and $c \in \mathbb{Q}$. Therefore we achieve by Theorem 7.2, (13) and (14) that $h(\tau) \in \mathcal{O}_1^{1\prime}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$. Hence the inclusion $\mathcal{O}_N^{1\prime}(\mathbb{Q}) \subseteq \mathcal{O}_1^{1\prime}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$ also holds. $\square$

REMARK 7.5. For $N \geq 2$, let Fr'_N be the set of *weak* Fricke families of level N, namely, the families $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ satisfying (F1), (F2′) and (F3). It is also a ring under the operations as in (5). In a similar way to the proof of Theorem 4.3, one can readily show that Fr'_N is isomorphic to ${\mathcal{O}^1_N}'(\mathbb{Q})$. Therefore, we deduce by Theorem 7.4 that a family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ is a weak Fricke family of level N if and only if there is a polynomial $P(x,y,z,w) \in \mathbb{Q}[x,y,z,w]$ so that

$$h_{\mathbf{v}}(\tau) = P(j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}, f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)) \quad \text{for all } \mathbf{v} \in \mathcal{V}_N.$$

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